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CORRECTION TO

ADMISSIBLE ESTIMATORS, RECURRENT DIFFUSIONS, AND INSOLUBLE BOUNDARY VALUE PROBLEMS

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In the above paper (*Ann. Math. Statist.* **42** 855–903) we introduced the diffusion $\{Z_t\}$ defined on E^m as having local mean $\nabla \log f^*$ and local variance 2I.

C. Srinivasan (private communication) has pointed out the following difficulty with this definition and our usage of $\{Z_t\}$. Since $\nabla \log f^*$ is a C^∞ function Z_t may always be defined locally (see e.g. McKean (1969)). However it may happen that $\{Z_t\}$ "explodes" in a finite time. To be precise, define the random time (time of explosion) as

$$\mathscr{T}^x = \sup_{R\to\infty} \inf\{t\colon Z_t^x \geqq R\}.$$

Either $\mathscr{T}^x = \infty$ w.p. 1, or not. In the latter case our definition of $\{Z_t\}$ is defective for $t \geqq \mathscr{T}$. To repair the definition let $Z_t = \infty$ if $t \geqq \mathscr{T}$. Z_t is then a well-defined diffusion on $E^m \cup \{\infty\}$.

Let us make some remarks to clarify the effect of this new definition of $\{Z_t\}$.

(1) *All of the main results of the paper remain true with this new definition of* $\{Z_t\}$ *exactly as they are stated in Brown* (1971); *except for Theorem* 4.3.1 *which requires a minor change*. (See (5, vi) below.) This includes all the results labeled as Theorems or Corollaries. However, some of the Lemmas must be modified. See below.

(2) If $\Pr\{\mathscr{T}_E^x < \infty\} > 0$ for some $x \in E^m$ then $\Pr\{\mathscr{T}^x < \infty\} > 0$ for all $x \in E^m$. In this case $\{Z_t\}$ is transient according to the definition (4.1.4) (which remains appropriate even with the above, revised definition of $\{Z_t\}$). These facts can be deduced from the discussion in McKean (1969, Section 4.4) and from previously described properties of $\{Z_t\}$.

(3) The situation $\Pr\{\mathscr{T}^x < \infty\} > 0$ is possible for diffusions of the type considered here; but only if $\sup\{f^*(x)\colon |x| \doteq r\}$ increases exceedingly rapidly as $r \to \infty$. As an example suppose $m = 1$ and $f^*(\{k\}) = e^{k^2/2}/k!$ $k = 0, 1, 2, \cdots$. Then $f^*(x) = \exp(e^x - x^2/2)$. Hence $\nabla \log f^*(x) = e^x - x$. It follows from Feller's test for explosion that $\Pr\{\mathscr{T}^x < \infty\} = 1$. (See e.g. McKean (1969, page 65).)

On the other hand, if $\limsup_{r\to\infty} \{||\nabla \log f^*(x)||\colon ||x|| = r\}/r < \infty$ then $\Pr\{\mathscr{T}^x < \infty\} = 0$, by Hasminskii's test (McKean (1969, page 102)). It can be shown that this is the case if $\int_{|x|<r} dF(x) = O(e^{kr})$ as $r \to \infty$ for some $k < \infty$.

(4) The only formal result in our paper which directly uses $\Pr\{\mathscr{T} < \infty\} = 0$ in its statement or proof is Lemma 4.2.1. Some later results use this Lemma in their proof but otherwise make no use of $\Pr\{\mathscr{T} < \infty\} = 0$. Lemma 4.2.1 remains correct if $m = 1$, or if $K = E^m$. This is easy to check. Therefore:

If $\Pr\{\mathscr{T} < \infty\} = 0$, *or* $m = 1$, *or* $K = E^m$ *all of the results in Brown* (1971) *are valid as stated. All of the proofs are also valid as stated except that some very minor modifications are needed in the proof of Lemma* 4.2.1. In particular, all results relating to the recurrent case are valid as given (since then $\Pr\{\mathscr{T} < \infty\} = 0$).

(5) We now sketch the modifications which are required in situations not covered by the above remark.

(i) A stronger statement is implied in the proof of Lemma 3.5.1 than is actually stated in the Lemma. (3.5.5) implies that there is a function $\alpha(r) \nearrow \infty$ as $r \to \infty$ ($\alpha(r) \geqq m$) such that

$$\lim_{r\to\infty} \sup_{\{x: |x| \geqq r, x \in K^{\alpha(r)}\}} (g^*(x)/f^*(x)) = 0 .$$

(ii) Define

$$K^* = \bigcup_{r\in(0,\infty)} \{x: |x| \geqq r, x \in K^{\alpha(r)}\} .$$

The statement in (i) then reads

(a) $$\lim_{r\to\infty} \sup_{\{x: |x| \geqq r, x \in K^*\}} (g^*(x)/f^*(x)) = 0 .$$

(iii) In Sections 4.2 and 4.3 the symbol K^m should generally be replaced by K^*. In particular the definition of the set O^R should be changed to read

$$O^R = \{x: |x| > 1, \text{ and } |x| < R \text{ or } x \notin K^*\} ,$$

and $T_R{}^x$ should be defined relative to this definition of O^R. ($T_R{}^x = \inf\{t: Z_t{}^x \notin O^R\}$.)

(iv) With these revised definitions the statement of Lemma 4.2.1 becomes correct.

Proof. Since $K^{\alpha(R)} \supset K^m$ the arguments on page 875 of Brown (1971) do show that for all x

(b) $$\Pr\{\exists t \ni Z_t{}^x \notin O^R \text{ or } \mathscr{T}^x < \infty\} = 1 .$$

This expression should replace (4.2.5).

(4.2.2) and (4.2.3) also imply that for each x

$$\lim_{R\to\infty} \inf\{t: d(Z_t{}^x\} > R\} = \infty \quad \text{w.p. } 1 .$$

Hence the random quantity $s^x = \sup\{d(Z_t{}^x): 0 \leqq t < \mathscr{T}^x\}$ satisfies $s^x < \infty$ almost everywhere on the set where $\mathscr{T}^x < \infty$. It follows that almost every sample path for which $\mathscr{T}^x < \infty$ must exit to explosion inside the set $K^{\alpha^{-1}}(s^x)$. In particular since $\alpha(r) \nearrow \infty$ almost every sample path for which $\mathscr{T}^x < \infty$ must enter all sets of the form $\{x: |x| > r, x \in K^{\alpha(r)}\}$ for sufficiently large values of r. Since O^R contains all such sets for $r > R$ it follows from (b) that, $\Pr\{T_R{}^x < \infty\} = 1$ which was to be proved.

(v) Lemma 4.2.2 is then correct with K^* substituted for K^m in its statement and proof.

(vi) The condition

(c) $$\lim_{r\to\infty} \sup_{\{x: x \in K^*, |x| \geqq r\}} j(x) = 0$$

should replace (4.1.2) in the definition of the set J. ((c) is related to (a) in the same way that (4.1.2) is related to the conclusion (3.5.3) of Lemma 3.5.1.) The statement and proof of Theorem 4.3.1 are then correct with J as defined from (c).

(vii) Similar notational changes are required in Section 5.2 which refers to Theorem 4.3.1 and the related material.

No other significant changes are required in the manuscript to allow for the possibility that $\{Z_t\}$ explodes in a finite time.

REFERENCES

BROWN, L. D. (1971). Admissible estimators, recurrent diffusions, and insoluble boundary value problems. *Ann. Math. Statist.* **42** 855–903.

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